Truth – non-additive measure for determination of relative density of sands using CPT measurements

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Relative density (D_r) is an important parameter in geomechanics. It indicates the state of density of a sandy soil and is used to estimate other engineering properties of soil. Several empirical correlations between D_r and CPT data are available in the literature. No single correlation, however, seems to be able to predict correctly D_r for all sands. For example, the correlation proposed by Villet et al. (1981) [9] is able to predict reliably D_r for sands of low compressibility. The correlation defined by Schmertmann (1978) [6] is more applicable to sands of high compressibility, while the correlation defined by Baldi et al. (1982) [1] was developed for sands of medium compressibility. In fact, compressibility of sands is not a well-defined parameter. A comprehensive model involving all the three correlations is difficult to develop. It is often more practical to first perform a calculation based on each correlation and then combine the results into a single overall result using linearly weighted average operator. This method is based on the assumption that the effects of evaluation of individual compressibilities are independent of one another and consequently are additive. However, the partial compressibilities are not orthogonal, and significant coupling exists among them. The relationship among the partial scores associated with different compressibilities can be quite complex; their effects are interactive. Thus, a simple linear combination of the partial correlations is incapable of capturing the noise and synergy of the information contained in these correlations; a highly non-linear process is required in its place. For this purpose, we introduce an idea of non-additive measures/truth measures based on multi-valued logic. Then, an aggregation operator using fuzzy integral will be used to determine the relative density of sands from CPT data.

CLASSICAL APPROACH FOR DETERMINING RELATIVE DENSITY

A general relationship, $D_r - q_c$, established by Kulhawy et al. (1991) [4] based on a database of 24 sands is represented as:

$$D_r^2 = \left(\frac{1}{Q_F}\right)\left[\frac{q_c / p_a}{(\sigma'_v / p_a)^{0.5}}\right] \tag{1}$$

where:

- p_a – atmospheric pressure,
- q_c – the cone-tip resistance,
- σ'_v – effective overburden stress,
- Q_F – an empirical constant determined by least-square regression analyses for normally consolidated (NC) sands of low, medium and high compressibility, respectively.

To characterize the sand compressibility, the friction ratio, r:

$$r = \frac{f_s}{q_c}\ [\%] \tag{2}$$

is usually used, where, f_s denotes the sleeve friction. To determine D_r, a weighted aggregation technique is developed in the paper presented by Juang et all (1996) [3] and used to combine the three base correlations in the form:

$$D_r = D_r^L W^L + D_r^M W^M + D_r^H W^H \tag{3}$$

where, D_r^k, $k = L, M, H$ are relative densities, defined by (1), depending on the correlations defined for sands of low, medium and high compressibility respectively through an empirical constant Q_F; W^k, denotes weights which are determined based on a „similarity" measure of three predefined levels of compressibility.

This technique is based on an implicit assumption that effects of the three compressibility levels (L, M, H) are viewed as additive $\{W^L + W^M + W^H = 1$ and $0 \le W^J \le 1\}$. This assumption is, however, not always reasonable as indicated by Viertl (1987) [8], Wang et al (1992) [10], Chi (2000) [2] and others.

TRUTH – NON-ADDITIVE MEASURE

First value of truth stated by true (T = 1) and then false ($\neg$T = 0) was introduced by Boole (1847). It is called two-valued (T, $\neg$T) logic. By same way, we can state the terms: „necessarily true", „possibly true" ($\Box$T, $\Diamond$T), in modal logic, by ($\Box$T, $\Diamond$T, $\neg\Diamond$T, $\neg\Box$T) which appear as four logic values. Similarly, we call different numbers between 0 and 1, in multi-valued logic ([0, 1]) – truth values (see Chi 2000 [2]), which in this work are called truth measures (τ). We can use truth values to express the degree of evidence, which may represent, for example, the degree of certainty, the degree of belief or the degree of important etc. of any object. Let X be a nonempty and finite set, $\aleph$ be a nonempty class of subsets of X; a truth measure on (X, $\aleph$) is a mapping $\tau: \aleph \to [0, 1]$, which really satisfies the following requirements:

a) $\tau(\varnothing) = 0$ and $\tau(X) = 1$ (boundary requirements) – on the one hand, the empty set does not contain any element, so obviously it cannot contain the element of our interest. On the other hand, the finite set X containing all elements under consideration must contain our element as well.

b) $E \in \aleph$, $F \in \aleph$ and $E \subset F$ imply $\tau(E) \le \tau(F)$ (monotonicity) – when we know with some degree of certainty that the element belongs to a set, then our belief that it belongs to a larger set containing the former set can be greater or equal, but it cannot be smaller.

τ, satisfying the above conditions (a, b), is called Lebesgue measure in the sense that for any Borel subset B,

$$\tau(A) = \tau(B \cup C) = \tau(B) \quad \text{for} \quad A = B \cup C, \tau(C) = 0 \tag{4}$$

It is called also a fuzzy measure in Sugeno's sense. This measure, with a loose additivity $\{\tau(E \cup F) = \tau(E) + \tau(F)$ for $E \cap F = \varnothing\}$, is considered to be a non-additive measure. Here, condition (b) (monotonicity) is substituted for the additive condition of the measure. It has a term with the combination of all elementary fuzzy measures multiplied by a factor λ, $\lambda > -1$.

$$\tau(E \cup F) = \tau(E) + \tau(F) + \lambda.\tau(E)\tau(F) \tag{5}$$

where, λ has an effect similar to a weight factor for interaction between the properties. Fuzzy measures satisfying mentioned condition is called as λ-fuzzy measure. If $\lambda = 0$ then τ can be used as a additive measure (probability measure). For a set of elements E_i, $E_i \in X$, relationship (5) can be used recursively and gives:

$$\tau\left(\bigcup_{i=1}^{n} E_i\right) = \frac{1}{\lambda}\left\{\prod_{i=1}^{n}\left[1+\lambda.\tau(E_i)\right]-1\right\};\ \lambda \neq 0 \tag{6}$$

As, $\tau(X) = 1$, when, $\bigcup_{i=1}^{n} E_i = X$ for a fixed set of $\{\tau^i\}$, $0 < \tau^i < 1$, we have:

$$\tau\left(\bigcup_{i=1}^{n} E_i\right) = 1 = \frac{1}{\lambda}\left\{\prod_{i=1}^{n}\left[1+\lambda.\tau(E_i)\right]-1\right\};\ \lambda \neq 0 \tag{7}$$

Then, the parameter λ will be obtained by solving the equation:

$$1+\lambda = \left\{\prod_{i=1}^{n}\left[1+\lambda.\tau(E_i)\right]\right\};\ \ \lambda \in (-1, \infty)\ \text{ and }\ \lambda \neq 0 \tag{8}$$

Note that this measure can be used to convey the expert's opinion of the situation on a scale with the truth dimension or the degree of importance indicating an uncertainty component in our knowledge.

FUZZY INTEGRALS

Let $(X, \wp)$ be a measurable space, where $X \in \wp$; $\wp$ is a σ-algebra of sets in the class of all finite subsets of X. A real-valued function f: $X \rightarrow (-\infty, \infty)$ on X is called as a measurable function if for any Borel set B:

$$f^{-1}(B) = \{x \mid f(x) \in B\} \in \wp \tag{9}$$

The functional relationship between measurable function, f, and fuzzy measure, τ, is represented by the Sugeno's integral as follows: let $X \in \wp$, $f \in \mathbf{F}$, $\mathbf{F}$ is the class of all finite nonnegative measurable functions defined on $(X, \wp)$. The fuzzy integral of $f(x)$ on X with respect to τ, which is denoted by $⨍ f(x)\mathrm{d}\tau$, is defined by:

$$⨍ f(x)\mathrm{d}\tau = \sup_{\alpha\in[0,\infty]}\left[\alpha \wedge \tau(X \cap \mathbf{F}_\alpha);\ \mathbf{F}_\alpha = \left\{x \mid f(x) \geq \alpha\right\}\right] \tag{10}$$

where:
$\mathbf{F}_\alpha$ – an α-cut of $f(.)$;
α – the threshold where the assumption is fulfilled, that the property in question is used in the minimal condition.

Let us look at an example presented in [10], We intend to evaluate three TV sets. We consider two quality factors: „picture" and „sound". These are denoted by $x1$ and $x2$ respectively, and the corresponding weights are w_i, $\Sigma w_i = 1$, $i = 1, 2$. An expert gives different scores, c_1, c_2, for each factor, $x1$ and $x2$ according to each TV set. Using the method of weighted mean we get synthetic evaluations of the three TV sets: $V_i = w_1c_1 + w_2c_2$. In the other way, we adopt now a fuzzy measure to characterize the importance of the two factors. For example, $\tau(\{x1\}) = 0.3$; $\tau(\{x2\}) = 0.1$; $\tau(X) = 1$; $X = \{x1, x2\}$ and $\tau(\varnothing) = 0$. Let us observe that this important measure, a truth measure, which is intuitively reasonable, is not additive: ($\tau\{x1, x2\} = 1 \neq \tau(\{x1\}) + \tau(\{x2\}) = 0,1 + 0,3 = 0,4$). Using fuzzy integral we can get synthetic evaluations of the three TV sets: $V_i^* = ⨍ f_i\mathrm{d}\tau$, where, f_i characterize the scores (c_i) given for three TV sets. The results obtained are represented in the table 1.

Table 1. Qualitative evaluation of three design variants

Variant	c_1	c_2	V $w_1 = 0.7$ $w_2 = 0.3$	V $w_1 = 0.4$ $w_2 = 0.6$	V^* $\tau\{x1\} = 0.3$ $\tau\{x2\} = 0.1$
1	1.0	0.0	**0.7**	0.4	0.3
2	0.0	1.0	0.3	**0.6**	0.1
3	0.5	0.5	0.5	0.5	**0.5**

According to our intuition, the third TV set should be identified as the best one among the three TV sets even though neither picture nor sound is perfect. Unfortunately, when using the method of weighted mean, no choice of the weights would lead to this expected result under the given scores. For example: (max $V_1 = 0.7 \rightarrow$ the first TV set is the best, although a TV set without any sound is not practical at all; or max $V_2 = 0.6 \rightarrow$ the second TV set is the best, even though a TV set with good sound but no picture is not a useful TV set. When using fuzzy integral we get a reasonable conclusion – the third TV set (max $V_i^* = 0.5 \rightarrow i = 3$) is the best, which agrees with our intuition.

NEW APPROACH FOR DETERMINING RELATIVE DENSITY

CPT data used for determining relative density are listed in the table 2:

Table 2. CPT data

CPT number	Depth [m]	σ'_v [kPa]	q_c [kPa]	f_s [kPa]	r_a [%]
12	6.0	81.0	5030	3	0.06

The „difference" measure of r_a and the predefined numbers, r_k, $k = L, M, H$ for the low, medium and high levels of compressibility respectively are defined as follows:

$$diff_{r_a}(k) = \left|r_a - r_k\right| \tag{11}$$

This distance is used as a means of measuring how close the actual friction ratio, r_a, is to each of the predefined numbers, r_k, according to different levels, k, of compressibility. Smaller distance indicates a higher degree of similarity. The compressibility measured by friction ratio corresponding to a higher similarity is assigned a greater value of truth, which is

$$\tau(k) = 1 - diff_{r_a}(k),\ \ k = L, M, H \tag{12}$$

i.e. sand, which is considered as sand having compressibility level k, $k = L, M, H$, is assigned the truth value $\tau(k)$.

According to Robertson and Campanella (1985) [5], the value r increases with increasing sand compressibility; for most normally consolidated (NC) sands, the predefined value of r for medium compressibility, r_M, is about 0,5%, but for sands of low compressibility, $r_L \approx 0\%$ and for sands of high compressibility, $r_H \approx 1\%$. Using these assumptions, the difference of the actual friction ratio $r_a = 0.06\%$ in comparison with the predefined numbers r_k, $k = L, M, H$ for different levels of compressibility is determined using equation (11). The truth, $\tau(k)$, assigned for the sand studied, which is considered as sand with compressibility levels L, M, H, respectively, will be determined by equation (12). We can obtain:

$$\tau(k) = \{0{,}94;\ 0{,}56;\ 0{,}06\},\ k = L, M, H$$

e.g. the sand with $r_a = 0.06$ is considered as sand having low compressibility with the assigned truth: $\tau(L) = 0.94$; medium compressibility with $\tau(M) = 0.56$ and high compressibility with $\tau(H) = 0.06$. Sands of the same mineral type could appear in different categories of compressibility depending on other factors, which are generally descriptive and not readily applicable for quantifying the compressibility (see Juang et. al. [3]). Then the expert's evaluations are needed. We support here evaluations by three experts, $\tau^*(k)$, based on both results mentioned, $\tau(k)$, and properties of the sand such as stress history, mineral type, particle angularity, particle size, particle surface roughness and others, which are:

Table 3. Qualitative evaluation of three experts according to low, medium and high compressibility of sand

Expert	$\tau^*(L)$ [–]	$\tau^*(M)$ [–]	$\tau^*(H)$ [–]
1	0.8	0.3	0.1
2	0.8	0.5	0.1
3	0.8	0.3	0.2

From that we can construct the λ-fuzzy-modal measure for all the other subsets of set X, $X = \{L \cup M \cup H\}$. Then, the λ-fuzzy measures for different subsets $\{(L \cup M), (L \cup H)$ and $(M \cup H)\}$ are defined by equation 8 and the truth of these subsets $\{\tau(L \cup M), \tau(L \cup H)$ and $\tau(M \cup H)\}$ are defined by equation (5). Next, value D_r for the sand with the actual friction ratio, r_a, is calculated using the Sugeno integral with $\alpha = \left\{D_r^L, D_r^M, D_r^H\right\}$, where, D_r^L, D_r^M, D_r^H are determined by equation (1) for sands of low, medium, and high compressibility respectively. It is represented as follows:

$$D_r^F = \int f \mathrm{d}\tau = \left[D_r^L \wedge \tau(X \cap F_{D_r^L})\right] \vee$$
$$\vee \left[D_r^M \wedge \tau(X \cap F_{D_r^M})\right] \vee \left[D_r^H \wedge \tau(X \cap F_{D_r^H})\right]$$

where:
„$\wedge$" and „$\vee$" – „min" and „max" operations respectively.

This fuzzy integral differs from the above weighted aggregation operator in that both objective evidence supplied by various sources $\left\{D_r^L, D_r^M, D_r^H\right\}$ and the expected worth of subsets of these sources $\left\{\tau(X \cap F_{D_r^L}), \tau(X \cap F_{D_r^M}), \tau(X \cap F_{D_r^H})\right\}$ are considered in the aggregated process. Here, it is worth noticing that the value obtained from comparing two quantities (D_r^k and τ) in terms of the „min" operator is interpreted as the grade of agreement between real possibilities and the expectation. The obtained results are shown in the table 4:

Table 4. Determination of relative density, D_r, by τ-fuzzy measure and fuzzy integral

Expert	$\tau^*(L)$ [–]	$\tau^*(M)$ [–]	$\tau^*(H)$ [–]	$\tau(L \cup M)$ [–]	$\tau(L \cup H)$ [–]	$\tau(M \cup H)$ [–]	D_r^F [%]
1	0.80	0.30	0.10	0.95	0.85	0.38	41.0
2	0.80	0.50	0.10	0.98	0.84	0.56	42.8
3	0.80	0.30	0.20	0.93	0.88	0.46	42.8

Let us notice that changes of results D_r^F depending on changes of $\{\tau^*(L), \tau^*(M), \tau^*(H), \tau(L \cup M), \tau(L \cup H), \tau(M \cup H)\}$ confirm the requirement that the relative truth of the compressibility should be taken into account in the fuzzy-integral operator. Finally, to reduce the influence of subjective biases of individual experts and to obtain a more reasonable evaluation, D_r^*, we can use an arithmetic average of the results obtained from three experts:

$$D_r^* = \overline{D_r^F} = \frac{1}{3}(0{,}41 + 0{,}428 + 0{,}428) = 0{,}422$$

The complete results (from objective evidence, D_r^k, the synthetic evaluation using weighted average approach, D_r^J and the synthetic evaluation using fuzzy-integration-based approach, D_r^*) are listed in the following table

Table 5. The complete results obtained from weighted average approach and fuzzy-integration-based approach

D_r^L [%]	D_r^M [%]	D_r^H [%]	D_r^J [%]	D_r^* [%]
41.0	42.8	44.8	41.0	42.2

SUMMARY OF CASE STUDY

Predicted values D_r^k, $k = L, M, H$ are calculated based on a set of three compressibility levels that are believed to be applicable to sands of low, medium and high compressibility, respectively, depending on the value of the friction ratio (r) that is influenced by mineral type of sands studied. However, as noticed earlier, sands of the same mineral type could be in different categories of compressibility.

Predicated value $D_r^J = D_r^L$, i.e. the result obtained, depends closely on the friction ratio ($r_a = 0.06 \approx 0$), which is determined without effects of the necessary qualitative factors. Besides, it is calculated based on the method of weighted mean, which is based on an implicit assumption that the compressibility levels: L, M, H are „independent" of one another, and their effects are viewed as additive. This, however, is not justifiable in some real problems.

Using a fuzzy measure/truth measure and using a fuzzy integral as a synthetic evaluator for determining the predicated value D_r^* can produce a satisfactory result.

CONCLUSION

If we have accepted a subjective property of geo-uncertainty then dealing with uncertainty means dealing with human ability. It is not only the question of the uncertainty quantification but also the elicitation and aggregation of human knowledge; i.e., dealing with uncertainties in respect of their relationship. Using the method mentioned above, the evidence – the CPT data at the classification level can be combined to obtain a partial evaluation for the relative density of sands, D_r at the compressibility level. Each of these levels has a different degree of importance/truth in the recognition of the classes. That is each compressibility level gives evidence supporting or rejecting an accurate and reliable result of D_r in the scene constrained by the fact that its identification is uncertain. Fuzzy integral with a non-additive measure allows us to take into account the relative important/truth of various compressibility levels, as well as the interactions of information contained in subsets of these levels. In this paper, we have focused on the practical problem – determining of D_r of sands using CPT data. We have shown that fuzzy measure with non-additive character and fuzzy integral possess advantages relative to other techniques for aggregating partial results from multiple information sources. It should also be helpful in many other applications that require effective and transparent combining of heterogeneous information sources.

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